Improvement of Structural Models Using Covariance Analysis and Nonlinear Generalized Least Squares

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The next generation of large, flexible space structures will be too light to support their own weight, requiring a system of structural supports for ground testing. The authors have proposed multiple boundary-condition testing (MBCT), using more than one support condition to reduce uncertainties associated with the supports. MBCT would revise the mass and stiffness matrix, analytically qualifying the structure for operation in space. The same procedure is applicable to other common test conditions, such as empty/loaded tanks and subsystem/system level tests. This paper examines three techniques for constructing the covariance matrix required by nonlinear generalized least squares (NGLS) to update structural models based on modal test data. The methods range from a complicated approach used to generate the simulation data (i.e., the correct answer) to a diagonal matrix based on only two constants. The results show that NGLS is very insensitive to assumptions about the covariance matrix, suggesting that a workable NGLS procedure is possible. The examples also indicate that the multiple boundary condition procedure more accurately reduces errors than individual boundary condition tests alone.

Nomenclature

$\{b\}$ = unknown vector
$[C]$ = correlation matrix
$[D]$ = diagonal standard deviation
$\{d\}$ = standard deviation vector
E = expected value operator
$\{e\}$ = error vector
f = analysis frequency, Hz
f^T = test frequency, Hz
g_{ii} = ground spring
Δg = ground spring uncertainty, $\epsilon_K g_{ii}$
$[ID]$ = cross orthogonality matrix
K = stiffness matrix
ΔK_{ij} = stiffness uncertainty, $\epsilon_K K_{ij}$
k = number of variables being estimated
M = mass matrix
ΔM = mass uncertainty, $\epsilon_M M_{ii}$
m = number of analysis modes
m^T = number of test modes
n = number of observations, $\Sigma(m_i + m_i n_i)$
n_i = degrees of freedom in the ith mode
s^2 = standard error
$[V]$ = covariance of $\{e\}$
$[V^T]$ = test covariance
var = variance operator
$[X]$ = observation matrix
$\{y\}$ = observation values
ϵ_f = estimated frequency error
ϵ_M = estimated mass error
ϵ_K = estimated stiffness error
ϵ_ϕ = estimated mode shape error
Δ = perturbation
ϕ = analytical mode shape
$\hat{\phi}$ = mass normalized analytical mode shape
$\hat{\phi}_*$ = normalization node mode shape value
ϕ^T = max normalized test mode shape
$\{\hat{\phi}^T\}$ = mass normalized test mode shape
λ = eigenvalue, $(2\pi f)^2$

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Introduction

SPACE structures have become so large and flexible that they cannot stand up in a 1-g field without a major system of supports. Such structures are designed through modal analysis and require preflight testing to qualify them for their deployed operational environment. Typically, there are stringent pointing or internal alignment requirements (e.g., as for a large antenna or interferometer) requiring predictable modal response to achieve the mission objectives. Unfortunately, the support system needed to restrain buckling on Earth makes it extremely difficult to ground test in the deployed condition.

Two approaches have been suggested for ground testing large space structures: multiple boundary-condition testing (MBCT) to update the mass and stiffness model,[1,2] and zero-gravity supports to minimize the influence of the support system.[3] It is possible that both techniques could be employed together, since they are complementary. Zero-gravity supports are designed to shift the frequency range of the support structure below that of the test structure. As the structure becomes larger, zero-gravity supports become increasingly impractical. Furthermore, the support system changes the local structure where it is attached.

The authors propose that, since the supports cannot be avoided, they should be *moved* to acquire more data, until sufficient data has been acquired to check the mathematical model of the structure. Support locations hide errors because the supports limit the motion used to reveal the errors. By moving the supports, some of the test data will expose each error. If zero-gravity supports are used, moving the boundary conditions should generate the same modal results. If moving the zero-gravity supports does not generate matching modal data, MBCT techniques can be used to resolve the differences.

MBCT will deal with an unusually large amount of data from various Earthbound test conditions. Not only must the analyst select the parameters that are in error but he must also weight the data to account for each parameter's sensitivity to different boundary conditions. Weighting is accomplished by constructing a covariance matrix in generalized least-squares

(GLS) theory. The GLS procedure can correct large errors in a structural model in the presence of numerous smaller errors modeled by the covariance matrix. The GLS procedure requires a covariance matrix, and Aitken's theorem states that the GLS estimator is the best linear, unbiased estimator (BLUE), assuming, of course, that the covariance matrix is correct.[4]

There are many ways of constructing a covariance matrix for use in the GLS procedure—all of which are based on *estimated* errors. The covariance matrix can be diagonal, amounting to a renormalization of the data. It can be a symmetric matrix calculated from model data. When a symmetric matrix is used, the off-diagonal terms account for correlation between the errors. The covariance matrix can even be estimated largely from test data. The advantage of test data is that it does not change as the iterative NGLS procedure creates new estimates for the model parameters.

The covariance matrix for a real MBCT problem will be very large. Some of the approaches presented here are probably too hard to calculate in the real situation. Still, the GLS procedure is very robust and capable of overcoming serious misspecification. It is likely that a dramatically simplified covariance matrix will still produce useful results.

To determine which procedure is best for MBCT, test data are simulated and the NGLS procedure is used to find large, known modeling errors. The simulation is as if the test beam had been constructed 30 different times and each beam had been tested in two different test configurations. All the sample errors are sized and correlated from the same known, correct covariance matrix (the analysis-based covariance matrix evaluated for the correct modal data). Then three methods of estimating the covariance matrix (diagonal, test based, and analysis based) are used in the NGLS procedure. Each of the two test configuration results and the MBCT results are calculated for all 30 test cases (one at a time) and the average and worst case answers are reported.

Nonlinear Generalized Least Squares Theory

The theoretical model is represented by a diagonal mass matrix $[M]$ and a symmetric stiffness matrix $[K]$. From this model a frequency vector $\{f\}$ and a set of mode shapes $[\phi]$ can be calculated. In modal testing, empirical modal data are obtained, $\{f^T\}$ and $[\phi^T]$, from which corrections are estimated for $[M]$ and $[K]$. Of course, the test data contains errors and so do the original $[M]$ and $[K]$ matrices. Any serious errors must be corrected against this background of numerous small errors. It is unrealistic to correct *all* the errors. That would require a very large amount of *errorless* test data—an impossibility. Instead, the analyst must concentrate on correcting the serious errors.

Least-squares curve fitting is widely used in engineering to solve this kind of problem. A Taylor series expansion is used to produce a suitable equation, and ΔM or ΔK are estimated, based on the errors in $\{f\}$ and $[\phi]$. This works well as long as data is restricted to either frequencies or mode shapes, but not both together. If data types with different units are used in simple least-squares curve fitting, the error equation has mixed units—it is not a valid engineering equation.

$$\text{error}(M,K)^2 = \sum^{\text{all } f^T} \left(\frac{f^T - f(M,K)}{\epsilon_f}\right)^2 + \sum^{\text{all } \phi^T} \left(\frac{\phi^T - \phi(M,K)}{\epsilon_\phi}\right)^2 \tag{1}$$

Some normalization is necessary for Eq. (1) to make physical sense. For a diagonal covariance matrix, the GLS procedure corrects the mixed units in the error equation by normalizing by the accuracy of each data point. A frequency is divided by the estimated size of the frequency error. A mode shape is divided by the estimated size of the mode shape error. This produces dimensionless errors where the numbers are large when the data is more significant. This works well with least squares since large numbers receive disproportionate, squared weighting. In Eq. (1) the error terms, ϵ_f and ϵ_ϕ, make the error dimensionless. Without ϵ_f and ϵ_ϕ, the two summations would have different units (e.g., frequency and translational mode shapes) and would not be correct. GLS must be used for the answer to be independent of the units of measure.

Using the least-squares approach, the analysis data in Eq. (1) is based on the first term of a Taylor series expansion of $\{f\}$ and $[\phi]$, in terms of the unknown model parameters, ΔK and ΔM. This is shown as follows:

$$f(K+\Delta K, M+\Delta M) - f(K,M) = \frac{\partial f}{\partial K}\Delta K + \frac{\partial f}{\partial M}\Delta M + O(\Delta^2) \tag{2}$$

$$\phi(K+\Delta K, M+\Delta M) - \phi(K,M) = \frac{\partial \phi}{\partial K}\Delta K + \frac{\partial \phi}{\partial M}\Delta M + O(\Delta^2) \tag{3}$$

The vector of unknowns $\{b\}$, the observation matrix $[X]$, and the vector of observation values $\{y\}$, are defined based on Eqs. (2) and (3). The perturbed values are viewed as test values, and the unperturbed values are viewed as analysis values. The higher-order terms are neglected and result in the procedure being nonlinear and iterative, thus it is nonlinear generalized least squares (NGLS).

$$\underset{k\times 1}{\{b\}} = \begin{Bmatrix} \Delta K \\ \Delta M \end{Bmatrix} \quad \Delta K \text{ and } \Delta M \tag{4}$$

are the same for all BC,

$$\underset{n\times 1}{\{y\}} = \begin{Bmatrix} f^T - f \\ \phi^T - \phi \\ \vdots \\ f^T - f \\ \phi^T - \phi \end{Bmatrix} \begin{matrix} \\ \\ \\ \\ \end{matrix} \begin{matrix} \text{first BC test and analysis} \\ \\ \\ \text{last BC test and analysis} \\ \\ \end{matrix} \tag{5}$$

Note: Mode shape data is concatenated one mode shape after another in a vector. Data from each boundary condition (BC) is sequentially concatenated.

$$\underset{n\times k}{[X]} = \begin{bmatrix} \partial f/\partial K & \partial f/\partial M \\ \partial \phi/\partial K & \partial \phi/\partial M \\ \vdots & \vdots \\ \partial f/\partial K & \partial f/\partial M \\ \partial \phi/\partial K & \partial \phi/\partial M \end{bmatrix} \begin{matrix} \text{first BC analysis} \\ \\ \\ \text{last BC analysis} \\ \\ \end{matrix} \tag{6}$$

For GLS, the answers $\{b\}$ are found as outlined below. First the statistical model will be presented along with the underlying assumptions about the covariance matrix. Then the solution to the problem is presented. Finally a few statistics are estimated, including the accuracy of the estimate for $\{b\}$.

Statistical model ($[X]$ and $\{y\}$ are known, $\{b\}$ and $\{e\}$ are stochastic and unknown):

$$\underset{n\times 1}{\{y\}} = \underset{n\times k}{[X]}\,\underset{k\times 1}{\{b\}} + \underset{n\times 1}{\{e\}} \tag{7}$$

The formal definition of the covariance matrix $[V]$ is shown in Eq. (8). For conventional least-squares, $[V]$ is assumed to be an identity matrix. A more useful equation for V will be pre-

sented later. ([V] is assumed to be known, $\{e\}$ and $E(s^2)$ are stochastic and unknown):

$$\underset{n\times n}{\text{var}[e]} = \underset{n\times 1}{E}\,\underset{1\times n}{(ee')} = s^2\underset{n\times n}{[V]} \tag{8}$$

The solution, $\{b\}$, ([X], $\{y\}$, and [V] known, solving for $E\{b\}$):

$$\underset{k\times 1}{\{b\}} = \left[\underset{k\times n}{[X]'}\,\underset{n\times n}{[V]^{-1}}\,\underset{n\times k}{[X]}\right]^{-1}\left\{\underset{k\times n}{[X]'}\,\underset{n\times n}{[V]^{-1}}\,\underset{n\times 1}{\{y\}}\right\} \tag{9}$$

It is useful to estimate a tolerance for the coefficients $\{b\}$. The standard deviation is typically used as a tolerance and is found by taking the square root of the diagonal of the covariance matrix for $\{b\}$. First, the value for s^2 from Eq. (8) is estimated as follows:

Standard error s^2 ([X], $\{y\}$, $E\{b\}$, and [V] known, solving for $E(s^2)$):

$$s^2 = \frac{1}{n-k}\left\{\underset{n\times 1}{\{y\}} - \underset{n\times k}{[X]}\,\underset{k\times 1}{\{b\}}\right\}'\underset{n\times n}{[V]^{-1}}\left\{\underset{n\times 1}{\{y\}} - \underset{n\times k}{[X]}\,\underset{k\times 1}{\{b\}}\right\} \tag{10}$$

The square root of the diagonal of var[b] is the tolerance estimate for $\{b\}$ (X, V, and $E(s^2)$ known, solving for var[b]):

$$\underset{k\times k}{\text{var}[b]} = s^2\left[\underset{k\times n}{[X]'}\,\underset{n\times n}{[V]^{-1}}\,\underset{n\times k}{[X]}\right] \tag{11}$$

Sensitivity Coefficients

The partial derivatives are known as sensitivity coefficients, and are available in closed form. Equations (12–15) are for mass normalized modes.[5] To change the normalization so that a specific node $\hat{\phi}_*$ is normalized, use the derivative of a ratio, i.e., $\partial(\hat{\phi}_{ij}/\hat{\phi}_*)/\partial K = (\partial\hat{\phi}_{ij}/\partial K)/\hat{\phi}_* - (\hat{\phi}_{ij}/\hat{\phi}_*^2)(\partial\hat{\phi}_*/\partial K)$:

$$\frac{\partial f_i}{\partial k_{rs}} = \frac{\hat{\phi}_{ri}\hat{\phi}_{si}}{8\pi^2 f_i} \tag{12}$$

$$\frac{\partial f_i}{\partial m_{rs}} = -\frac{f_i\hat{\phi}_{ri}\hat{\phi}_{si}}{2} \tag{13}$$

$$\frac{\partial\hat{\phi}_{ij}}{\partial k_{rs}} = \sum_{q=1}^{m}\begin{cases}\dfrac{\hat{\phi}_{iq}\hat{\phi}_{rq}\hat{\phi}_{sj}}{\lambda_j-\lambda_q} & j\neq q\\ 0 & j=q\end{cases} \tag{14}$$

$$\frac{\partial\hat{\phi}_{ij}}{\partial m_{rs}} = \sum_{q=1}^{m}\begin{cases}-\dfrac{\lambda_j\hat{\phi}_{iq}\hat{\phi}_{rq}\hat{\phi}_{sj}}{\lambda_j-\lambda_q} & j\neq q\\ -\dfrac{\hat{\phi}_{ij}\hat{\phi}_{rj}\hat{\phi}_{sj}}{2} & j=q\end{cases} \tag{15}$$

There are other formulations of the sensitivity coefficients available, with advantages and disadvantages to each formulation.[6]

Covariance Matrix

To implement NGLS, it is necessary to create a matrix [V] to estimate the covariance of the error $\{e\}$ in Eq. (8). The error is everything left over after the systematic term $[X]\{b\}$ is removed from $\{y\}$. One way to consider $\{e\}$ is as the random errors in $\{y\}$ that are obstructing a clear knowledge of $\{b\}$. [V] will be composed of test errors and also analysis errors in the construction of [K] and [M]. The higher-order terms in the Taylor series will be removed by successive iteration of the systematic term and will not contribute to [V]. [V] is the sum of the squares of the standard deviations of the test errors, [M] errors, and [K] errors, which are all assumed to be uncorrelated, normally distributed errors.

The covariance matrix is easily constructed, as long as the errors are either uncorrelated or completely correlated. If the errors are completely correlated, all the correlated errors are treated together as if they were just one error, i.e., one $\{d\}$ as follows:

$$\underset{n\times n}{[V]} = \sum_{\substack{\text{all}\\ \text{errors}}}\underset{n\times 1}{\{d\}}\,\underset{1\times n}{\{d\}'} \tag{16}$$

When the analyst estimates a cross-sectional area for a structural member, the shop will build it to some tolerance. The difference between the actual cross section and the nominal cross section is one of the small analysis errors in the stiffness matrix. This kind of error exists throughout the analysis data and potentially confuses the much larger mistakes that NGLS seeks to correct. The small errors are highly correlated in the modal data—each error can affect every frequency and mode shape.

A huge amount of tolerance data would potentially be needed to estimate the size of all such analytic errors in the mass and stiffness matrix. Accumulating large amounts of tolerance data is unnecessary because the central limit theorem suggests that the precise details of the distributions of the errors will not have much effect on the final covariance matrix. Instead, the covariance matrix can be estimated by assuming that all the stiffness terms have a fixed percentage error, $\epsilon_K = 5\%$. The mass matrix has similarly been assumed to have a percentage error, $\epsilon_M = 1\%$, on all the mass terms.

To estimate the individual terms in Eq. (16), the sensitivity coefficients from Eqs. (12–15) are multiplied by the error for each term from the mass and stiffness matrix. Every nonzero term of the mass and stiffness matrix must be considered separately to form a vector $\{d\}$ and must be added to the sum in Eq. (16).

In the stiffness matrix, the errors for four terms are completely correlated: K_{ii}, $-K_{ij}$, $-K_{ji}$, and K_{jj}. If an error occurs in any one of these terms, it will also occur in the other three because the four terms represent one physical spring in the analytical model. Thus, each stiffness error consists of four terms with the size of ΔK taken as $\epsilon_k K_{ij}$ and applied to all four terms. A separate uncorrelated error for the diagonal K_{ii} is also included, if there is a spring to ground at that point in the structure.

Errors in the mass and stiffness matrices can be correlated with one another. For example, an error in the cross-sectional area of a structural member will cause an error in both the mass and stiffness matrices. This kind of error was not considered because mass errors usually are corrected with ad hoc additions of weight to adjust the mass to the measured value.

Finally, the test data also contains small measurement errors. Test errors are important because they are not correlated with any other error source. Thus, there are as many independent test errors as there are test measurements. The test errors guarantee that the covariance matrix is not rank deficient and can be inverted as required in Eq. (9).

The test frequency data are assumed to be accurate to a constant measurement accuracy $\epsilon_f = 0.05$ Hz. The frequency error is assumed to be the same for all modes. The test mode shape data are assumed to be measured at a single level of response. To extract modes at a 1-g level, all the accelerometers are assumed to be calibrated to the same level, and the largest response is raised to 1 g. In the analysis, the maximum for a mode is normalized to one, and all the data for that mode is assumed to be accurate to $\epsilon_\phi = 0.05$ for a 5% error.

This assumption about ϵ_ϕ is the reason that the maximum value normalization is found to be advantageous. The assumptions used here are consistent with modal-dwell testing. They are not appropriate for the random or for the transient approaches to modal testing.

To predict the coefficients $\{b\}$ using Eq. (9), the inverse of the covariance matrix must be available at each iteration step. If the covariance matrix is based on analysis data, the estimated covariance matrix changes at each step in the NGLS process as the parameters ΔK and ΔM change.

This is not a trivial matter. The covariance matrix is much larger than any of the modal models. For example, if there

were 200 accelerometers used for mode shape data, and 15 modes were extracted for each of two boundary conditions, the covariance matrix would be $(200-1)\times 15\times 2$ ϕ terms (skipping one for normalization) plus 15×2 (f terms), giving a 6000×6000 matrix to be constructed and inverted at each step in the process. The rest of the calculation involving 200 degree-of-freedom eigenvalue solutions is insignificant by comparison.

A method must be found that is better than constructing and inverting such a huge matrix at every nonlinear step. The NGLS process is very robust and capable of overcoming misspecification. It is likely that a dramatically simplified covariance matrix (e.g., a diagonal) would still produce useful results.

Suppose that one used the covariance matrix for the test data alone, as shown in Eq. (17). The ϵ_f and ϵ_ϕ would be based on accuracy estimates for the test setup. Note that $[V^T]$ is diagonal, and because it is based strictly on test data, it does not change between solution steps. Furthermore, it is consistent with the sense of Eq. (7). If the parameter variations $\{b\}$ completely correct the model, all that would remain would be test errors. One problem with this approach is that the test frequency error tends to be small, so a great deal of weight (i.e., $[V^{-1}]$) is placed on the frequency data. It may be advisable to increase the frequency error ϵ_f to reflect the uncertainty associated with small modeling errors, especially errors in the stiffness matrix.

$$[V]_{n\times n} = \underset{\substack{n\times n\\ \text{test errors}}}{[V^T]} + \sum^{M_{ii}} \Delta M_{ii}^2 \underbrace{\begin{Bmatrix} \frac{\partial f}{\partial M} \\ \frac{\partial \phi}{\partial M} \\ \vdots \\ \frac{\partial f}{\partial M} \\ \frac{\partial \phi}{\partial M} \end{Bmatrix}}_{n\times 1} \underbrace{\begin{Bmatrix} \frac{\partial f}{\partial M} \\ \frac{\partial \phi}{\partial M} \\ \vdots \\ \frac{\partial f}{\partial M} \\ \frac{\partial \phi}{\partial M} \end{Bmatrix}'}_{1\times n} \text{analytic mass} + \sum^{K_{ii}} \Delta g^2 \underbrace{\begin{Bmatrix} \frac{\partial f}{\partial K_{ii}} \\ \frac{\partial \phi}{\partial K_{ii}} \\ \vdots \\ \frac{\partial f}{\partial K_{ii}} \\ \frac{\partial \phi}{\partial K_{ii}} \end{Bmatrix}}_{n\times 1} \underbrace{\begin{Bmatrix} \frac{\partial f}{\partial K_{ii}} \\ \frac{\partial \phi}{\partial K_{ii}} \\ \vdots \\ \frac{\partial f}{\partial K_{ii}} \\ \frac{\partial \phi}{\partial K_{ii}} \end{Bmatrix}'}_{1\times n} \text{analytic grounds}$$

$$+ \sum_{i>j}^{K_{ij}} \Delta K_{ij}^2 \underbrace{\begin{Bmatrix} \frac{\partial f}{\partial K_{ii}} - \frac{\partial f}{\partial K_{ij}} - \frac{\partial f}{\partial K_{ji}} + \frac{\partial f}{\partial K_{jj}} \\ \frac{\partial \phi}{\partial K_{ii}} - \frac{\partial \phi}{\partial K_{ij}} - \frac{\partial \phi}{\partial K_{ji}} + \frac{\partial \phi}{\partial K_{jj}} \\ \vdots \\ \frac{\partial f}{\partial K_{ii}} - \frac{\partial f}{\partial K_{ij}} - \frac{\partial f}{\partial K_{ji}} + \frac{\partial f}{\partial K_{jj}} \\ \frac{\partial \phi}{\partial K_{ii}} - \frac{\partial \phi}{\partial K_{ij}} - \frac{\partial \phi}{\partial K_{ji}} + \frac{\partial \phi}{\partial K_{jj}} \end{Bmatrix}}_{n\times 1} \underbrace{\begin{Bmatrix} \frac{\partial f}{\partial K_{ii}} - \frac{\partial f}{\partial K_{ij}} - \frac{\partial f}{\partial K_{ji}} + \frac{\partial f}{\partial K_{jj}} \\ \frac{\partial \phi}{\partial K_{ii}} - \frac{\partial \phi}{\partial K_{ij}} - \frac{\partial \phi}{\partial K_{ji}} + \frac{\partial \phi}{\partial K_{jj}} \\ \vdots \\ \frac{\partial f}{\partial K_{ii}} - \frac{\partial f}{\partial K_{ij}} - \frac{\partial f}{\partial K_{ji}} + \frac{\partial f}{\partial K_{jj}} \\ \frac{\partial \phi}{\partial K_{ii}} - \frac{\partial \phi}{\partial K_{ij}} - \frac{\partial \phi}{\partial K_{ji}} + \frac{\partial \phi}{\partial K_{jj}} \end{Bmatrix}'}_{1\times n} \text{analytic stiffness} \tag{18}$$

$$[V^T]_{n\times n} = \begin{bmatrix} \epsilon_f^2 & 0 & \cdots & 0 & 0 \\ 0 & \epsilon_\phi^2 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \epsilon_f^2 & 0 \\ 0 & 0 & \cdots & 0 & \epsilon_\phi^2 \end{bmatrix} \begin{matrix} \\ \text{first BC test} \\ \\ \\ \text{last BC test} \end{matrix} \tag{17}$$

For a diagonal covariance matrix with the proposed error levels ($\epsilon_f = 0.05$ Hz and $\epsilon_\phi = 0.05$) and the mode shape data normalized on the maximum value location for the test data, the NGLS procedure becomes identical to the least-squares procedure. Examination of Eq. (17) shows that the diagonal covariance matrix is simply $0.05^2 = 0.0025$ times an identity matrix—a least-squares case where s^2 is multiplied by 0.0025. Using least-squares (as opposed to GLS) guarantees that the diagonal covariance procedure will have the smallest possible error $\{e\}$ from Eq. (7). GLS guarantees that var$[b]$ from Eq. (11) is as small as possible. Of course, when $[V]$ really is an identity matrix, both minimums occur simultaneously.

Equation (17) illustrates the covariance matrix for modes normalized on the maximum value location from the test data. To produce the same curve fit results using mass normalization, the diagonal cases are not NLS because each mode's maximum response, $\hat{\phi}_*$, is included in the covariance matrix as $(\epsilon_\phi/\hat{\phi}_*)^2$. The value of $\hat{\phi}_*$ varies from mode to mode, so the covariance matrix is not an identity matrix.

A second approach is to estimate the covariance matrix with the initial analysis data as shown in Eq. (18), invert it, and use this matrix throughout the rest of the data reduction without updating. Then, when the process has converged, the covariance matrix is estimated again to obtain adequate statistical information about the estimates. It is likely that this approach will produce adequate answers, if the analysis model is reasonably close to the test data. By not updating the covariance matrix at each step, the computer time involved is reduced by the number of nonlinear iteration steps (e.g., typically six iteration steps are used in the examples so the computer run time is reduced to 2/6 of the time it would take to update at every step).

A third approach is to base the derivatives in Eq. (18) on test modal parameters, and analytic mass and stiffness parameters for the ΔM and ΔK terms. The derivatives from Eqs. (12–15) can be estimated with test data, though the sums for the derivatives cannot be done very accurately, and the mass normalization for the derivatives involves the theoretical mass matrix. The test data do not change, so the covariance matrix does not need to be updated if it is calculated this way. It is helpful to exclude the terms being fit, since all the variance in these parameters will be eliminated by using Eq. (9).

Correspondence Between Test and Analysis Modes

To use any least squares procedure, the test modes must be identified with the analysis modes. The easiest approach is to use cross orthogonality of the modes. First, each test mode is renormalized, based on the analytic mass matrix. Then, the cross-orthogonality matrix is calculated as shown in Eq. (19).

$$[ID] = [\hat{\phi}^T]' [M] [\hat{\phi}] \tag{19}$$
$$m^T \times m \quad m^T \times n_i \; n_i \times n_i \; n_i \times m$$

Each term in the $[ID]$ matrix whose absolute value is larger than a threshold value (e.g., 0.6) indicates correspondence between the test and analysis mode. If two terms are larger than the threshold value, the largest is used. If the threshold value is sufficiently large, when modes are difficult to recognize, the test mode will not be identified and will be excluded until later in the procedure, when the modes should be more distinct.

Using the analytic mass matrix to renormalize the test mode shapes correlates the errors in the analytic mass matrix with the errors in the test mode shape data. Thus, later analysis should normalize both the test and analysis modes to 1 at the location of the test mode's largest value. Normalizing at the maximum test mode shape location means that all the mode shape errors are the same value, ϵ_ϕ. The covariance matrix can be constructed and inverted without reference to the modal identification.

Constraints

Another problem is the tendency to produce estimates that are physically unrealizable. Occasionally a value of ΔK will be estimated that causes a negative spring in the analysis model. To correct this problem, calculate $\{b\}$ using Eq. (9), and then limit $\{b\}$ so it can never produce negative stiffness terms. This can be done using Lagrangian multipliers.[7] A good constraint value to use is $\Delta K \geq -K/2$.

A simpler equivalent procedure is to perform the calculations without constraints and check the results to see if any of the constraints have been violated. If any ΔK is smaller than $-K/2$, the row that calculates ΔK in Eq. (9) is replaced by the equation $\Delta K = -K/2$. This is done by inserting a row with a single 1 at the position of ΔK in the $[X'][V^{-1}][X]$ term and the value $-K/2$ in the $[X'][V^{-1}]\{y\}$ term. All violated constraints are eliminated in this way. Using constraints ensures that the algorithm will never produce a negative stiffness or decrease a stiffness by an unreasonably large amount.

Step Sizes

To estimate values for $\{b\}$ in Eq. (7), a set of test and analysis data is prepared, a covariance matrix is constructed and inverted, and finally Eq. (9) is used to generate a set of perturbation estimates for the next point to be evaluated. This process is repeated until it converges.

Run time is shortened if the number of steps is reduced, but the process may diverge. If the number of steps is increased and smaller steps are taken, the process will converge more often. For example, if the number of steps was three, it would help the convergence to move a third of the way to the answer on the first step, half the way to the answer on the second step, and all the way to the answer on the third step. This approach dramatically reduces the possibility of overshooting the actual answer. In the immediate vicinity of the answer, NGLS converges rapidly, so a few extra steps should be taken at the end to refine the result.

Statistics and Convergence Criterion

There is a simple statistical test to see whether the process has converged. For a linear model, the ratio between any of the estimated parameters $\{b\}$ and the estimated standard deviation for that parameter, the square root of the diagonal of var$[b]$ from Eq. (11), can be shown to be distributed according to Student's t distribution.

To determine whether the process has converged, the parameter estimate $\{b\}$ is calculated along with its variance var$[b]$. Then the ratio of the parameters to the square root of the diagonal of var$[b]$ is calculated. If any of these numbers exceeds 0.05, another iteration step should be taken (subject to some upper limit on the number of steps taken).

The iteration continues until all the answers are insignificant. A cumulative total of the values of $\{b\}$ is kept as a final reportable value for $\{b\}$. The ratio of this cumulative parameter to the standard deviation computed at the end of the process makes the most sense as the estimated t statistic for the variable. This t statistic can be used to decide if the coefficient is statistically different from zero.

Table 1 Ideal mass and stiffness properties

Stiffness, lb/in.	Masses, lb_m
10- 20: 30.0	10- 20: 0.433
20- 30: 60.0	20- 30: 0.433
30- 40: 42.0	30- 40: 0.433
40- 50: 60.0	40- 50: 0.433
50- 60: 60.0	50- 60: 0.433
60- 70: 120.0	60- 70: 0.433
70-130: 60.0	70-130: 0.433

Numerical Example

Consider a one-dimensional beam in bending, as shown in Fig. 1. The problem is to reconcile data from testing the beam in more than one configuration with analytical models of each test configuration containing systematic errors. The suspected location of the errors has been identified through engineering judgment. NGLS is used to quantify the size of the errors, and through interpretation of the statistics, to confirm or deny the location of the errors.

The ideal mass and stiffness associated with beam are shown in Table 1.

Two test conditions are available: a test with the beam simply supported at each end and a cantilevered beam for the nodes from 10 to 70.

The first test configuration is shown in Fig. 1. If the test configuration is ideal, this test produces five modes below 50 Hz, at 8.8, 18.6, 26.3, 35.6, and 42.6 Hz. Ground springs are at node 20 = 30.0 lb/in. and node 120 = 60 lb/in.

The second test configuration is shown in Fig. 2. The ideal test configuration produces three modes below 50 Hz, at 9.6, 26.7, and 38.7 Hz. There is one ground spring for test configuration two at node 60 = 120.0 lb/in.

Correct Covariance Matrix

Assembly of the results into a covariance matrix produces an 81×81 symmetric matrix with five rows for the frequencies of the first model, 5×11 rows for the mode shapes of the first model, three rows for the frequencies of the second model, and 3×6 rows for the mode shapes of the second model. The normalization node for each mode shape is included, so test error in measuring the normalization point will be considered.

To form the correct covariance matrix, the value of each spring in Table 1 is perturbed by 5%. For each spring, four sensitivity coefficients are summed, and 5% of the nominal spring rate is used as a perturbation value, e.g., $\Delta K_{ij} = 0.05 K_{ij}$. These perturbation values are used in the analytic stiffness part of Eq. (18) to calculate the contribution of that one spring to the overall covariance. This procedure is repeated for all 12 springs, some of which become ground springs or are missing, depending on the configuration. The

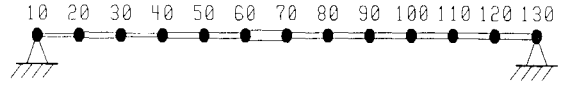

Fig. 1 Test configuration one.

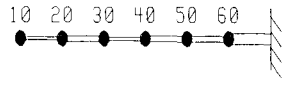

Fig. 2 Test configuration two.

Table 2 Analytic mass and stiffness properties

Stiffness, lb/in.	Masses, lb_m
10-130: 60	10-130: 0.433

Table 3 Summary for model 1

	1	2	3	4
	Diagonal covariance matrix			
Avg b:	−29.6±1.5	−17.3±2.8	−0.3± 6.3	58.8± 15.3
SD b:	1.2±0.4	0.9±0.4	8.9± 6.1	76.3± 76.0
Min b:	−32.7±2.4	−22.5±0.9	−11.2± 5.0	37.8± 22.6
Max b:	−27.0±1.4	−11.6±2.3	19.0±21.1	98.9±416.0
Signif:	30	30	2	0
$s^2=2.612$, $rms_f=0.119$ Hz, $rms_\phi=7.24\%$, 6.6 steps				
	Test-based covariance matrix			
Avg b:	−29.7±1.2	−17.6±2.0	0.2± 5.1	60.0± 10.3
SD b:	1.1±0.3	1.2±0.4	9.2± 5.3	90.9± 48.3
Min b:	−33.2±1.4	−21.5±0.7	−7.0± 4.3	40.5± 61.3
Max b:	−26.9±1.1	−13.1±2.5	12.6±29.8	82.8±274.5
Signif:	30	30	0	0
$s^2=1.597$, $rms_f=0.240$ Hz, $rms_\phi=6.72\%$, 6.1 steps				
	Analysis-based covariance matrix			
Avg b:	−29.8±1.2	−17.6±1.9	0.4± 4.6	59.0± 9.0
SD b:	4.1±1.0	7.2±2.2	19.5− 7.3	114.9± 48.8
Min b:	−33.2±6.2	−21.6±4.0	−7.5±17.3	39.4± 45.3
Max b:	−27.2±3.7	−14.1±6.3	10.6±23.4	74.2±119.8
Signif:	30	23	0	0
$s^2=1.347$, $rms_f=0.220$ Hz, $rms_\phi=6.79\%$, 6.2 steps				

only difference with ground springs is that only one sensitivity coefficient is used, instead of four. Springs that are missing in a configuration contribute nothing.

Similarly, all 13 masses are perturbed by 1%, one at a time, and sensitivity coefficients are used to calculate the contribution of the analytic mass to the overall covariance, as shown in Eq. (18). Where masses are located at a ground or are missing, they contribute nothing.

Finally, the contribution of the test error is added to the diagonal of the covariance matrix. The frequency error of 0.05 Hz is squared and added to the diagonal of the frequency data. The mode shapes are normalized by their maximum value; thus, an error of 5% in the shapes contributes $0.05^2=0.0025$ to the diagonal of all the mode-shape data.

Building Test and Analysis Data

The test data is constructed by calculating the eigenvalues and eigenvectors of the correct covariance matrix. Each eigenvector is scaled by the square root of its eigenvalue in a manner very reminiscent of mass normalized normal modes. Next, a Gaussian random number generator is used to scale each standard deviation vector of the covariance matrix. Then, the whole system of 81 values is reassembled, using Eq. (16) as one of the 30 samples used for test data in the analysis. This process is related to the statistical procedure known as factor analysis.[8] Factor analysis offers a number of ways of rotating and adjusting the errors so they are more interpretable.

To produce analysis data, models are built of the two cases with a consistent set of errors in them. For this example, it was assumed that the analyst had modeled the beam as uniform, as shown in Table 2.

NGLS Results

In Tables 3-5, the same 30 samples of test data have been analyzed using the three methods of calculating the covariance matrix. Four variables were fit into each model:

$$\text{var 1 } \Delta K_{10,20} = -30.0 \text{ lb/in.}$$

Table 4 Summary for model 2

	1	2	3	4
	Diagonal covariance matrix			
Avg b:	−29.8±1.5	−18.2±2.4	1.0± 4.7	65.1± 12.1
SD b:	0.2±0.1	2.5±1.1	9.8± 6.1	31.2± 18.6
Min b:	−32.7±0.2	−23.4±1.1	−7.1± 4.3	47.3± 20.2
Max b:	−26.6±0.2	−14.2±1.9	14.3±14.7	89.1± 74.5
Signif:	30	30	0	19
$s^2=1.35$, $rms_f=0.0246$ Hz, $rms_\phi=5.50\%$, 6.0 steps				
	Test-based covariance matrix			
Avg b:	−29.9±1.5	−18.0±2.5	0.9± 4.8	64.0± 11.0
SD b:	0.3±0.1	2.8±1.2	15.0±10.2	102.7± 52.6
Min b:	−32.7±0.3	−23.3±1.3	−6.2±12.8	44.7± 60.5
Max b:	−26.4±0.3	−13.5±2.5	14.4±19.9	85.1±128.7
Signif:	30	30	0	1
$s^2=1.20$ $rms_f=0.0966$ Hz, $rms_\phi=5.22\%$, 5.8 steps				
	Analysis-based covariance matrix			
Avg b:	−29.9±1.5	−18.0±2.5	1.0± 4.8	63.3± 11.1
SD b:	3.0±1.2	7.9±3.2	25.0±13.1	141.3± 62.1
Min b:	−32.7±3.0	−23.2±4.0	−6.4±15.0	43.0± 92.8
Max b:	−26.4±3.6	−13.5±7.1	14.1±29.7	81.6±142.7
Signif:	30	17	0	1
$s^2=1.19$, $rms_f=0.1000$ Hz, $rms_\phi=5.20\%$, 6.2 steps				

Table 5 Summary for both models together

	1	2	3	4
	Diagonal covariance matrix			
Avg b:	−29.9±1.5	−17.6±2.9	0.3± 4.6	61.7± 8.7
SD b:	0.2±0.1	0.7±0.3	4.0± 2.0	24.4±11.1
Min b:	−32.9±0.3	−22.8±0.7	−11.0± 2.8	48.4±15.2
Max b:	−27.0±0.2	−11.0±1.6	11.0± 6.9	77.8±35.7
Signif:	30	30	2	28
$s^2=2.58$, $rms_f=0.126$ Hz, $rms_\phi=6.99\%$, 6.7 steps				
	Test-based covariance matrix			
Avg b:	−29.9±1.3	−17.7±1.9	0.5± 3.8	61.5± 8.4
SD b:	0.2±0.0	0.8±0.2	4.4± 1.8	34.4±11.9
Min b:	−33.0±0.2	−21.7±0.7	−4.8± 3.9	46.7±24.4
Max b:	−26.9±0.2	−14.4±1.2	10.6±10.1	81.1±64.2
Signif:	30	30	0	13
$s^2=1.65$, $rms_f=0.181$ Hz, $rms_\phi=6.59\%$, 5.4 steps				
	Analysis-based covariance matrix			
Avg b:	−29.9±1.3	−17.7±1.9	0.6± 3.9	60.5± 7.4
SD b-:	3.2±0.8	6.8±2.0	15.8± 4.9	75.7±31.3
Min b:	−32.8±4.4	−21.8±3.5	−4.8± 9.9	45.6±50.2
Max b:	−26.8±3.1	−14.2±8.9	11.1±29.0	74.1±77.3
Signif:	30	24	0	0
$s^2=1.33$, $rms_f=0.191$ Hz, $rms_\phi=6.60\%$, 6.6 steps				

$$\text{var 2 } \Delta K_{30,40} = -18.0 \text{ lb/in.}$$

$$\text{var 3 } \Delta K_{40,50} = 0.0 \text{ lb/in.}$$

$$\text{var 4 } \Delta K_{60,70} = 60.0 \text{ lb/in.}$$

Given test data drawn from a known theoretical covariance matrix and analysis data with errors, it is possible to use NGLS to find the analysis errors using the theory presented earlier.

NGLS results from three covariance matrix procedures are compared: 1) a diagonal covariance matrix shown in Eq. (17) that depends on two estimated errors: a frequency accuracy estimate and a mode-shape accuracy tolerance; 2) a symmetric covariance matrix shown in Eq. (18) based on test modal data (for the derivatives) and analytic mass/stiffness data, so the data going into the model does not change as the analysis progresses; and 3) a symmetric covariance matrix shown in Eq. (18) based on analytic data (the covariance matrix is generated at the first step and updated when the mode identification changes and on the last step).

The correct covariance matrix is usually not known, so some estimate of it must be made. The analysis-based covariance 3 is the most accurate (in these examples, the construction is

identical to the covariance matrix used to generate the data—it is the correct answer), but it is difficult to calculate and must be updated as the analysis progresses. The test-data based covariance 2 avoids the update issue by using the test data to calculate the covariance, but the accuracy of the mode-shape derivatives are suspect since a complete set of modes is unavailable. The diagonal covariance matrix 1 avoids the update issue by using only test data errors and is very simple to calculate, but it misses all the correlation information conveyed by the full matrix.

The average estimated value for the four parameters is shown in the row labeled "Avg b," and the computed standard deviation based on the 30 samples is shown as the ± value in the same row.

For comparison, the column labeled "SD b" gives the average standard deviation based on the square root of the diagonal of Eq. (11). Since the procedure is nonlinear, differences between the two estimates are expected. Standard deviations of this estimate based on the 30 samples are shown as the ± value in the same row.

The two estimates labeled "Min b" and "Max b" are the largest and smallest estimate for each of the variables from all the samples. The ± values are the Eq. (11) results that went with the case. Of course, all the maximum and minimum values of b do not occur in the same sample. The row labeled "Signif" indicates how many of the 30 cases produced statistically significant results, i.e., had a t statistic >2.

The three error sums at the end of each table give the average standard error from Eq. (10) (s^2), the average rms frequency error (rms_f), and the average rms mode shape error (rms_ϕ). The covariance matrix is an estimate of the actual error and has not been normalized; thus the value of s^2 can be interpreted as the square of the ratio between the observed errors and the predicted errors. The two rms numbers indicate how much error exists between the test data and the analysis data after the corrections have been made.

The tables reveal that all three routines did a very good job in producing the correct estimate for the parameters. This is slightly misleading, however. Because 30 samples were used, if the averages did not converge to the correct answer, it would indicate the presence of bias someplace in the process. Since every effort has been made to keep the process free from bias, the averages should be accurate.

Unfortunately in real applications, 30 samples are not available to work with. Typically only one sample is available with no information as to how far it is from correct other than the statistics calculated for that case. The standard deviations and maximum and minimum cases indicate the range of answers the routines can produce in the presence of small errors.

One problem in all three tables is that variable 3, which was zero, shows up with fairly large values (11.2–19.0 in Table 3, for the diagonal covariance matrix). This shows the need for computing the variance of the estimate to see whether a variable is significantly different from zero. In the largest case, 19.0 ± 21.1, the ratio of the estimated coefficient to the tolerance does not exceed 2.0, thus the coefficient is not statistically significant. Still, for the diagonal covariance case, two examples show up in the maximum and minimum values (Table 3 and Table 5) where a coefficient is incorrectly reported as statistically significant when actually it is not. Four coefficients were erroneously reported as statistically significant in all three tables.

The other problem in all three tables is the difficulty calculating the statistics for variable 4. Here one of the stiffness values was increased by a factor of two, but all three procedures have difficulty finding the error. The reason for this is that an increase in stiffness does not have as large an influence on the mode shapes as a decrease in stiffness. Beyond some level of stiffness the procedure has difficulty identifying the amount that the stiffness was increased. The stiff element does not bend very much, so there is little information in the modal test data to indicate how stiff the element really is.

In general, for MBCT the procedures worked better when the two models were merged. In particular, the range on variable 4 is tighter for the maximum and minimum cases, and the estimates for the variable 3 are not so far off. This is not surprising, in general; having more data is good. Still, these improvements are not large. The single model cases (Tables 3 and 4) reduce the overall error nearly to the level that was input in the simulation. The overall error is measured by the rms_f and rms_ϕ values. It is difficult to improve on these results in the multiple boundary-condition case (Table 5). The single model analyses do a good job of correcting the data, leaving little for the MBCT analysis to improve upon.

The diagonal covariance matrix based on estimated test errors is adequate except for the statistics. In Table 5, the statistics based on the diagonal covariance matrix represent the known situation in the test cases more accurately than any other procedure. However, the diagonal covariance procedure tends to find the zero variable, variable 3, statistically significant. In statistics this is known as a type-1 error. A type-1 error is not surprising since the statistics are set up to have 1 chance in 20 of producing this error. However, in some cases, the zero variable is erroneously reported as having a high probability of being nonzero ($t = 11.0/2.8 = 3.9$, or only about 1 chance in 5000 of being zero). When the assumptions about the error were made more accurate and reflected the analysis errors as well as the test errors, the diagonal covariance procedure produced many more type-1errors and was also much less convergent.

In earlier work where the modes were mass normalized, the opposite trends were observed: A reasonable error estimate produced good results, while a test-error-based estimate produced poor results. This lack of robustness is disturbing, and suggests that the diagonal covariance approach may occasionally produce poor results. It would be advisable to include some soft spring as a variable that the analyst is confident is not wrong to check for type-1 errors in the procedure. Some fine tuning of the error assumptions (i.e., the ratio between frequency and mode shape errors) may be necessary. Still, all estimates were unbiased over all runs made, even with the diagonal covariance matrix when it was not working very well.

A test-data-based covariance matrix offers the cheapest alternative to improve the statistics. Since the test data does not change with the iteration steps, it does not need to be recalculated once the data has converged. The statistics produced from this procedure are almost as accurate as the diagonal covariance matrix but tend to fail to identify a known increased stiffness as statistically significant (i.e., a type-2 error). This is a difficult case, since the stiffness is twice the surrounding structure's stiffness and produces little change in the measured modal data. The procedure produced type-2 errors on 57% of the samples.

The analysis-data-based covariance matrix updated at the end of the analysis required twice as many calculations of the test-data-based covariance matrix. It produced results that were almost the same as when the covariance matrix was updated at every single step, but required much less computer effort. The covariance matrix from which the sample data was generated was almost identical to the final covariance matrix estimate using this procedure. There is a much worse problem with type-2 errors using the analysis data procedure than for either of the previous procedures. All the estimates associated with variable number 4 are found to be statistically insignificant.

Remembering that the analysis-data-based covariance matrix is the correct covariance matrix, the right answer using linear statistics is that the perturbation for variable 4 is not statistically significant. The change is too small to be reliably distinguished from zero. Using the variance of the estimate based on the 30 samples, the coefficient clearly is statistically significant. This kind of conflicting answer is common when using linear curve fit procedures as a local approximation to a nonlinear curve.

Overall, the results summarized above are consistent with the error assumptions used. Test error was very significant, and consequently the covariance matrix was fairly diagonal, especially for the mode shape data. The correlation coefficients between mode shapes were less than 25% and tended to approach zero. Between frequencies and mode shapes, the correlation coefficients were less than 50%. Between frequencies, the correlation coefficients were as high as 70%. If the mode shape data were more accurately measured, e.g., 1% errors instead of 5% errors; the off-diagonal mode shape terms of the covariance matrix would be more important. This also explains why the test-data-based covariance matrix worked well. The frequency derivatives based on test data are quite accurate, whereas the less important mode shape derivatives are relatively inaccurate.

Conclusions

Generalized least-squares analysis should be used when both frequency and mode shape data are used together. Using conventional least squares (NLS), the results will not be independent of the units of measure, because the error equation does not have consistent units. Changing to radians/second instead of Hertz will change the answer, not just the units in which the answer is reported. The difference between NGLS and NLS is subtle. The diagonal covariance matrix cases reported here actually are NLS because the frequency error, 0.05 Hz, and the mode shape error, 0.05 (a ratio), are the same number. As a general rule, for data with mixed units, always use NGLS. If by chance $[V]$ turns out to be an identity matrix, the result will be calculated correctly by an NGLS procedure.

NGLS requires a covariance matrix but the procedure is very insensitive to the construction of the covariance matrix. Any reasonable estimate for $[V]$ produces unbiased estimates for the coefficients (i.e., the average result from the 30 samples closely approximated the correct answer in all cases). This occurs even in the presence of small errors associated with test measurements and analysis/manufacturing flaws that were not removed by the procedure. This conclusion is significant because it is difficult to estimate the covariance matrix accurately. In this paper, the correct covariance matrix was based on four parameters: ϵ_K, ϵ_M, ϵ_f, and ϵ_ϕ. Many more parameters would be required if NGLS were sensitive to the covariance matrix.

All three covariance matrix procedures clearly work and successfully reduce the data from this small problem. However, the analysis-based covariance approach (the correct answer in this simulation) is probably too computationally intensive to be practical. It took 2 h of VAX computer time to calculate the 30 cases presented here. The diagonal covariance approach is much faster (30 min on the VAX for the 30 cases); however, the ratio of frequency errors to mode shape errors that produced good results was not the correct ratio that was input in the simulation. The lack of robustness suggests that some ad hoc manipulation maybe required when using the diagonal covariance approach. The test-data-based covariance matrix took 1 h of VAX time to calculate the 30 cases, and offers a simple option that probably will be more robust on large problems.

Still, computer time to invert the covariance matrix is likely to be a problem except with the diagonal covariance approach. Based on the results found here, measurement error dominates the mode shape errors (e.g., the mode shapes could be treated as a diagonal composed of measurement errors), and only the frequency errors need to be treated as a full matrix calculated from test data. Computer time could be dramatically reduced with a specialized inverse routine set up to take advantage of this geometry.

These conclusions only apply to sine dwell testing—other covariance matrices are appropriate for the random or transient methods of modal testing. The covariance matrix is based on estimated errors. A detailed error analysis should be done for the other two methods to estimate a covariance matrix for them.

In an MBCT situation, where artificial supports are necessary for ground testing, the following conclusions are warranted, based on this investigation. If data from only one set of support conditions are used, it will be difficult to guarantee that all the parameters significant to the flight situation have been uncovered. It is possible for an error such as the fourth variable in the example to be missed as a "statistically insignificant" result. The probability of correctly identifying these errors is improved, but not eliminated, by taking more data from other boundary conditions. The improvement apparently is caused by the increased quantity of data rather than by any inherent advantage from the MBCT technique. More work is needed to identify which boundary conditions are required to find all possible errors. It seems likely that a more complicated structure would require tests conducted with more than one boundary condition.

The MBCT procedure needs a full-scale demonstration test to validate the overall concept and the data reduction procedures outlined here. There are several testbeds being constructed. We hope to arrange a demonstration on one of them soon using real hardware and unknown modeling errors.

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